

# Geometry of Lie integrability by quadratures

J.F. Cariñena<sup>†a)</sup>, F. Falceto<sup>‡b)</sup>, J. Grabowski<sup>◇c)</sup>, and M.F. Rañada<sup>†d)</sup>

<sup>†</sup> *Departamento de Física Teórica and IUMA, Facultad de Ciencias  
Universidad de Zaragoza, 50009 Zaragoza, Spain*

<sup>‡</sup> *Departamento de Física Teórica and BIFI, Facultad de Ciencias  
Universidad de Zaragoza, 50009 Zaragoza, Spain*

<sup>◇</sup> *Polish Academy of Sciences, Institute of Mathematics,  
Sniadeckich 8, PO Box 21, 00-656 Warsaw, Poland*

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## Abstract

In this paper we extend the Lie theory of integration in two different ways. First we consider a finite dimensional Lie algebra of vector fields and discuss the most general conditions under which the integral curves of one of the fields can be obtained by quadratures in a prescribed way. It turns out that the conditions can be expressed in a purely algebraic way. In a second step we generalize the construction to the case in which we substitute the Lie algebra of vector fields by a module (generalized distribution). We obtain much larger class of integrable systems replacing standard concepts of solvable (or nilpotent) Lie algebra with distributional solvability (nilpotency).

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<sup>a)</sup> *E-mail address:* jfc@unizar.es

<sup>b)</sup> *E-mail address:* falceto@unizar.es

<sup>b)</sup> *E-mail address:* jgrab@impan.pl

<sup>c)</sup> *E-mail address:* mfran@unizar.es

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## 1 Introduction

Integrability of a given system of differential equations is a recurrent subject of very much interest and it has been an active field of research along the last years. The meaning of integrability, however, is not well-defined and has a different sense within each theory, and it is only rigourously defined in each specific field. Of course integrability means that you can find the general solution in an algorithmic way, but for instance you can restrict yourself to search for solutions of a previously selected class of functions, polynomial functions, rational functions, etc. The existence of additional structures, for instance compatible symplectic structures, may be useful. The possible dependence of integrability of a lucky choice of coordinates is also a relevant point.

The objective of this work is to reanalyze the classical problem of integrability by quadratures, without any recourse to the existence of additional compatible structures, but using modern tools of algebra and geometry. The first relevant result is due to Lie and we aim to slightly generalize the result of his work. Partial integrability is related to the existence of some first integrals and infinitesimal symmetries, but in the generic case we may not have enough number of them for an effective finding of the complete solution and for most part of physical systems we will be unable to answer about integrability or non-integrability of the system. This gives even more importance to results about characterization of particular cases in which such a question can be answered. This has motivated reinvention of many integration techniques that had been previously introduced by distinguished mathematicians of past centuries. We will fix our attention on Lie approach to the problem which was based on the use of Lie algebras of symmetry vector fields, and more in particular solvable Lie algebras [1]. For a recent description of other related integrability approaches see e.g. [2].

After proving a generalization of the classical result by Lie, motivated by some natural examples which do not fit into Lie's scheme, we develop new ideas in which Lie algebras of vector fields are replaced by certain modules of vector fields (distributions). The corresponding concept of *distributional integrability* allows for much larger class of examples

and potential applications. Our approach and the corresponding results are, up to our knowledge, novel and original.

The paper is organized as follows: Section 2 is devoted to introduce notation and establishing the relation of integrability by quadratures with the standard Arnold-Liouville integrability. In particular we recall the classical theorem by Lie on integrability by quadratures whose proof, for the simplest case  $n = 2$ , is given. In section 3 we recall some concepts of cohomology needed to analyze the existence of solutions for a system of differential equations. A lemma establishing in cohomological terms necessary conditions for the existence of solution of a first order system of differential equations is given. In section 4 we introduce an iterative process for solving a system of first order differential equations, expressed in geometrical terms as a vector field  $\Gamma$  on a manifold  $M$ . The procedure consists in constructing by quadratures a sequence of nested Lie subalgebras  $L_{\Gamma,k}$  of a  $\dim(M)$ -dimensional Lie algebra of vector fields  $L$ , such that any of them contain the dynamical vector field  $\Gamma$ . If in some step the resulting Lie algebra is Abelian, we can obtain with one more quadrature the general solution. Some interesting algebraic properties are studied in Section 5, and in particular we prove the in order for the process, outlined above, to work the Lie algebra must be solvable. Conversely, it is proved that when a solvable Lie algebra  $L$  contains an Abelian ideal  $A$ , then  $(M, L, \Gamma)$  is Lie integrable, i.e. the previous algorithm works, for any  $\Gamma \in A$ . A particular case is the theorem by Lie discussed in section 2. We also prove that if the Lie algebra  $L$  is nilpotent, then  $(M, L, \Gamma)$  is Lie integrable for any  $\Gamma \in L$ .

An interesting example which has been recently studied from a Hamiltonian viewpoint is reanalyzed in section 6 without any recourse to the symplectic structure of the phase space, but focusing our attention on the Lie algebra structure of the symmetries. Last section is devoted to extending the previous results to the more general situation in which, instead of having a Lie algebra  $L$  of vector fields, we have a vector space  $V$  such that its elements do not close a finite dimensional real Lie algebra, but they generate a general integrable distribution of vector fields. We develop new geometric approach to integrability based on certain algebraic properties of distributions. The introduced concept of *distributional integrability* and the corresponding version of Lie's theorem provides a large new class of systems integrable by quadratures. We also prove in this context results similar to that of Section 5, using original ideas of *distributional solvability* and *distributional nilpotency*.

## 2 Integrability by quadratures

Recall that an autonomous system of differential equations,

$$\dot{x}^i = f^i(x^1, \dots, x^n), \quad i = 1, \dots, n, \quad (1)$$

is geometrically interpreted in terms of a vector field  $\Gamma$  in a  $n$ -dimensional manifold  $M$  with a local expression

$$\Gamma = \sum_{i=1}^n f^i(x^1, \dots, x^n) \partial_i, \quad \partial_i \equiv \frac{\partial}{\partial x^i}. \quad (2)$$

The integral curves of  $\Gamma$  are the solutions of (1). Integrating the system amounts to determine its general solution. In particular, integrability by quadratures means that

you can determine the solutions by means of a finite number of algebraic operations and integrals of known functions.

The two main techniques for solving the system are the determination of first-integrals and the search for infinitesimal symmetries of the vector field. The set of first-integrals provides us a foliation such that the vector field is tangent to the leaves and reducing in this way the problem to a family of lower dimensional ones, one in each leaf, while the knowledge of symmetries of the vector field, suggests us to use adapted coordinates, the system decoupling then into lower dimensional subsystems.

More specifically, if  $F_1, \dots, F_r$ , are functionally independent, i.e. such that  $dF_1 \wedge \dots \wedge dF_r \neq 0$ , first-integrals, then for a given a vector field  $\Gamma \in \mathfrak{X}(M)$ , we can consider the foliation whose leaves are the level sets of the function of rank  $r$ ,  $(F_1, \dots, F_r) : M \rightarrow \mathbb{R}^r$ , and as  $\Gamma$  is tangent to each leave, the problem is reduced to that of the vector fields  $\tilde{\Gamma}_c$  defined in each  $n - r$  dimensional leave  $M_c = \mathbf{F}^{-1}(c)$ ,  $c \in \mathbb{R}^r$ . Of course, the best situation is when  $r = n - 1$  because then the leaves to be considered are one-dimensional, giving us the solutions to the problem, up to a reparametrization.

The other way of reducing the problem is based on the knowledge of infinitesimal (or one-parameter subgroups of) symmetries, i.e. vector fields  $X$  such that  $[X, \Gamma] = 0$ . The result of the Straightening out Theorem [3] asserts the existence of adapted coordinates  $(y^1, \dots, y^n)$  in a neighbourhood of a point where  $X$  is different from zero, i.e. such that

$$X = \frac{\partial}{\partial y^n} ,$$

and its integral curves are obtained by solving a subsystem involving only the other  $n - 1$  coordinates. Note however that the new coordinates  $y^1, \dots, y^{n-1}$ , are constants of motion and therefore we cannot find easily such coordinates in a general case. Moreover, the information provided by two different symmetry vector fields cannot be used simultaneously unless they commute.

It is clear that the if we use such rectifying coordinates for  $\Gamma$  the integration is immediate, the solution being

$$y^k(t) = y_0^k, \quad k = 1, \dots, n - 1, \quad y^n(t) = y_0^n + t.$$

This proves that the concept of integrability by quadratures depends on the choice of initial coordinates, because using these adapted coordinates the system is always integrable by quadratures.

Both constants of motion and infinitesimal symmetries can be used simultaneously if some compatibility conditions are satisfied. We can say that a system admitting  $r < n - 1$  functionally independent constants of motion is integrable when we know furthermore  $s$  infinitesimal symmetries  $X_1, \dots, X_s$ , with  $r + s = n$  such that

$$[X_a, X_b] = 0, \quad a, b = 1, \dots, s, \quad \text{and} \quad X_a F_\alpha = 0, \quad \forall a = 1, \dots, s, \quad \alpha = 1, \dots, r.$$

The constants of motion determine a  $n - r$  foliation and the former condition means that the restriction of vector fields  $X_a$  to the leaves are tangent to such leaves.

Sometimes we have additional geometric structures that are compatible with the dynamics [4]. For instance a  $2m$ -dimensional manifold  $M$  is endowed with a symplectic

structure  $\omega$ . Such 2-form relates, by contraction, in a one-to-one way vector fields and 1-forms, and vector fields  $X_F$  associated with exact 1-forms  $dF$  are said to be Hamiltonian vector fields. Compatible here means that the dynamical vector field itself is a Hamiltonian vector field  $X_H$ . The particularly interesting case of Arnold–Liouville definition of (Abelian) complete integrability [5, 6, 7, 8] is a particular case where  $r = m$ , the vector fields are  $X_a = X_{F_a}$  and, for instance,  $F_1 = H$ . The regular Poisson bracket defined by  $\omega$ , i.e.  $\{F_1, F_2\} = X_{F_2}F_1$ , allows us to express the above tangency conditions as

$$X_{F_b}F_a = \{F_a, F_b\} = 0, \quad a, b = 1, \dots, m,$$

i.e. the  $m$  functions are constants of motion in involution and the corresponding Hamiltonian vector fields commute.

Our aim in this paper is to study integrability in absence of additional compatible structures, the main tool being the symmetries of the given vector field very much in the approach started by Lie. We will see that if the given vector field is part of an appropriate Lie algebra of vector fields, then it is integrable by quadratures in any chart.

Given a vector field  $\Gamma \in \mathfrak{X}(M)$  in a differentiable manifold  $M$ , the set of strict symmetries of  $\Gamma$  is a linear space  $\mathfrak{X}_\Gamma(M) = \{X \in \mathfrak{X}(M) \mid [X, \Gamma] = 0\}$ . Obviously,  $\Gamma \in \mathfrak{X}_\Gamma(M)$ . The flow of a vector field  $X \in \mathfrak{X}_\Gamma$  preserves the set of integral curves of the dynamical vector field  $\Gamma$ . There are vector fields generating flows preserving the set of integral curves of  $\Gamma$  up to a reparametrization, those of the set  $\mathfrak{X}^\Gamma(M) = \{X \in \mathfrak{X}(M) \mid [X, \Gamma] = f_X \Gamma\}$ , where  $f_X \in C^\infty(M)$ . It is also a real linear space containing  $\mathfrak{X}_\Gamma(M)$ . Actually, vector fields in  $\mathfrak{X}^\Gamma(M)$  preserve the one-dimensional distribution generated by  $\Gamma$ .

Notice that  $\mathfrak{X}_\Gamma(M)$  is a Lie subalgebra of the real Lie algebra  $\mathfrak{X}^\Gamma(M)$ , as a consequence of Jacobi identity, but is not an ideal.

The problem of integrability by quadratures depends on the determination by quadratures of the necessary first-integrals and on finding adapted coordinates, or, in another words, finding a sufficient number of tensor invariants [9, 10]. The first result is due to Lie who established the following theorem:

**Theorem 1.** If  $n$  vector fields  $X_1, \dots, X_n$ , which are linearly independent at each point of an open set  $U \subset \mathbb{R}^n$ , generate a solvable Lie algebra and are such that  $[X_1, X_i] = \lambda_i X_1$  with  $\lambda_i \in \mathbb{R}$ , then the differential equation  $\dot{x}_i = X_1 x_i$  is solvable by quadratures in  $U$ .

*Proof.* We only prove here the simplest case  $n = 2$  (the higher dimensional one is a particular case of our Theorem 3). Then, the derived algebra is one-dimensional and therefore the Lie algebra is solvable. The differential equation can be integrated if we are able to find a first integral  $F$  for  $X_1$ , i.e.  $X_1 F = 0$ , such that  $dF \neq 0$  in  $U$ . In this case we can implicitly define one variable, for instance  $x_2$ , in terms of the other one by  $F(x_1, \phi(x_1)) = k$ , and the differential equation determining the integral curves of  $X_1$  is in separate variables, i.e. integrable by quadratures.

Let  $X_1$  and  $X_2$  be two vector fields such that  $[X_1, X_2] = \lambda_2 X_1$ . note that since  $n = 2$  there exists a 1-form  $\alpha_0$ , which is defined up to multiplication by a function, such that  $i(X_1)\alpha_0 = 0$ . Obviously as  $X_2$  is linearly independent of  $X_1$  at each point,  $i(X_2)\alpha_0 \neq 0$ . The 1-form  $\alpha = (i(X_2)\alpha_0)^{-1}\alpha_0$  satisfies the condition  $i(X_2)\alpha = 1$ , by definition, together with  $i(X_1)\alpha = 0$ , and we can see that  $\alpha$  is then closed, because  $X_1$  and  $X_2$  generate  $\mathfrak{X}(\mathbb{R}^2)$  and

$$d\alpha(X_1, X_2) = X_1\alpha(X_2) - X_2\alpha(X_1) + \alpha([X_1, X_2]) = \alpha([X_1, X_2]) = \lambda_2 \alpha(X_1) = 0.$$

Therefore, there exists, at least locally, a function  $F$  such that  $\alpha = dF$ . In other words, the function  $(i(X_2)\alpha_0)^{-1}$  is an integrating function for  $\alpha_0$ . The condition  $i(X_1)\alpha = 0$  means that  $F$  is a first integral  $F$  for  $X_1$ .

The (locally defined) function  $F$  such that

$$F(x^1, x^2) = \int_{\gamma(x^1, x^2)} \alpha,$$

where  $\gamma(x^1, x^2)$  is any curve joining a reference point  $(x_0^1, x_0^2) \in U$  with the point  $(x^1, x^2)$ , is the function we were looking for.  $\square$

We do not give the proof for  $n > 2$  because a more general result will be proved later on.

### 3 Local integration by quadratures of closed forms

Let us first recall some useful notions of Lie algebra cohomology. Let  $\mathfrak{g}$  be a Lie algebra and  $\mathfrak{a}$  a  $\mathfrak{g}$ -module respectively. In other words,  $\mathfrak{a}$  is a linear space that is the carrier space for a linear representation  $\Psi$  of  $\mathfrak{g}$ , i.e.  $\Psi : \mathfrak{g} \rightarrow \text{End } \mathfrak{a}$  satisfies

$$\Psi(a)\Psi(b) - \Psi(b)\Psi(a) = \Psi([a, b]), \quad \forall a, b \in \mathfrak{g}.$$

By a  $k$ -cochain we mean a  $k$ -linear alternating mapping from  $\mathfrak{g} \times \cdots \times \mathfrak{g}$  ( $k$  times) into  $\mathfrak{a}$ . We denote by  $C^k(\mathfrak{g}, \mathfrak{a})$  the space of  $k$ -cochains. For every  $k \in \mathbb{N}$  we define  $\delta_k : C^k(\mathfrak{g}, \mathfrak{a}) \rightarrow C^{k+1}(\mathfrak{g}, \mathfrak{a})$  by [11, 12]

$$\begin{aligned} (\delta_k \alpha)(a_1, \dots, a_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \Psi(a_i) \alpha(a_1, \dots, \widehat{a}_i, \dots, a_{k+1}) + \\ &+ \sum_{i < j} (-1)^{i+j} \alpha([a_i, a_j], a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_{k+1}), \end{aligned}$$

where  $\widehat{a}_i$  denotes, as usual, that the element  $a_i$  is omitted.

The linear maps  $\delta_k$  can be shown to satisfy  $\delta_{k+1} \circ \delta_k = 0$ . The linear operator  $\delta$  on  $C(\mathfrak{g}, \mathfrak{a}) = \bigoplus_{n=0}^{\infty} C^n(\mathfrak{g}, \mathfrak{a})$  whose restriction to each  $C^k(\mathfrak{g}, \mathfrak{a})$  is  $\delta_k$ , satisfies  $\delta^2 = 0$ . We will then denote

$$\begin{aligned} B^k(\mathfrak{g}, \mathfrak{a}) &= \{\alpha \in C^k(\mathfrak{g}, \mathfrak{a}) \mid \exists \beta \in C^{k-1}(\mathfrak{g}, \mathfrak{a}) \text{ such that } \alpha = \delta\beta\} = \text{Image } \delta_{k-1}, \\ Z^k(\mathfrak{g}, \mathfrak{a}) &= \{\alpha \in C^k(\mathfrak{g}, \mathfrak{a}) \mid \delta\alpha = 0\} = \ker \delta_k. \end{aligned}$$

The elements of  $Z^k(\mathfrak{g}, \mathfrak{a})$  are called  $k$ -cocycles, and those of  $B^k(\mathfrak{g}, \mathfrak{a})$  are called  $k$ -coboundaries. Since  $\delta^2 = 0$ , we have  $B^k(\mathfrak{g}, \mathfrak{a}) \subset Z^k(\mathfrak{g}, \mathfrak{a})$ . The  $k$ -th cohomology group  $H^k(\mathfrak{g}, \mathfrak{a})$  is defined as

$$H^k(\mathfrak{g}, \mathfrak{a}) := \frac{Z^k(\mathfrak{g}, \mathfrak{a})}{B^k(\mathfrak{g}, \mathfrak{a})},$$

and we will define  $B^0(\mathfrak{g}, \mathfrak{a}) = 0$ , by convention.

An interesting example is that of  $\mathfrak{g}$  being a finite-dimensional Lie subalgebra of  $\mathfrak{X}(M)$  (the space of vector fields on a manifold  $M$ ),  $\mathfrak{a} = \bigwedge^p(M)$  (the space of  $p$ -forms on  $M$ ), and the action given by  $\Psi(X)\zeta = \mathcal{L}_X\zeta$ . For instance the case  $p = 0$  has been used in [13] in the study of weakly invariant differential equations and  $p = 1, 2$ , play an interesting role in mechanics (see e.g. [12]). We will use next the particular case  $p = 0$ . In this case the elements of  $Z^1(\mathfrak{g}, \bigwedge^0(M) = C^\infty(M))$  are linear maps  $h : \mathfrak{g} \rightarrow C^\infty(M)$  satisfying

$$\mathcal{L}_X h(Y) - \mathcal{L}_Y h(X) = h([X, Y]) , \quad X, Y \in \mathfrak{X}(M),$$

and those of  $B^1(\mathfrak{g}, C^\infty(M))$  are those  $h$  for which  $\exists g \in C^\infty(M)$  with

$$h(X) = \mathcal{L}_X g .$$

At this point we first present the following preliminary lemma.

**Lemma 2.** Let  $\{X_1, \dots, X_n\}$  be a set of  $n$  vector fields that are linearly independent at each point of a  $n$ -dimensional manifold  $M$ . Then:

- 1) The necessary and sufficient condition for the system of equations for  $f \in C^\infty(M)$

$$X_i f = h_i, \quad h_i \in C^\infty(M), \quad i = 1, \dots, n, \quad (3)$$

to have a solution is that the 1-form  $\alpha \in \bigwedge^1(M)$  such that  $\alpha(X_i) = h_i$  be an exact 1-form.

- 2) If the previous  $n$  vector fields generate a  $n$ -dimensional real Lie algebra  $\mathfrak{g}$ , i.e. there exist real numbers  $c_{ij}{}^k$  such that  $[X_i, X_j] = \sum_k c_{ij}{}^k X_k$ , then the necessary condition for the system of equations to have a solution is that the  $\mathbb{R}$ -linear function  $h : \mathfrak{g} \rightarrow C^\infty(M)$  defined by  $h(X_i) = h_i$  is a cochain that is a cocycle.

*Proof.* 1) A function  $f$  is a solution of (3) if and only if  $\alpha = df$ .

- 2) Consider  $\mathfrak{a} = C^\infty(M)$  and the cochain determined by the linear map  $h$ . Now the necessary condition for the existence of the solution is written as:

$$X_i(h_j) - X_j(h_i) - \sum_k c_{ij}{}^k h_k = (\delta_1 h)(X_i, X_j) = 0$$

which is just the 1-cocycle condition. □

Most properties of differential equations are of a local character, and closed forms are locally exact, therefore we can restrict ourselves to appropriate open subsets  $U$  of  $M$ , i.e. open submanifolds, where every closed 1-form is exact, for instance assuming  $U$  to be arc-wise connected and simply connected. Then if  $\alpha$  is closed it is locally exact,  $\alpha = df$  in a certain open  $U$ ,  $f \in C^\infty(U)$ , and the solution of the system can be found by one quadrature: the solution function  $f$  is given by the quadrature

$$f(x) = \int_{\gamma_x} \alpha, \quad (4)$$

where  $\gamma_x$  is any path joining some reference point  $x_0 \in U$  with  $x \in U$ . We also remark that  $\alpha$  is exact,  $\alpha = df$ , if and only if  $\alpha(X_i) = df(X_i) = X_i f = h_i$ , i.e.  $h$  is a coboundary,  $h = \delta f$ .

In the particular case of the functions  $h_i$  appearing in the system being constant the condition for the existence of local solution reduces to  $\alpha([X, Y]) = 0$ , for each pair of elements,  $X$  and  $Y$  in  $\mathfrak{g}$ , i.e.  $\alpha$  vanishes on the derived Lie algebra  $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ .

## 4 A generalization of the Lie theory of integration.

Let us consider a family of  $n$  vector fields,  $X_1, \dots, X_n$ , defined on a  $n$ -dimensional manifold  $M$ . We assume that they close a Lie algebra  $L$  over the real numbers

$$[X_i, X_j] = \sum_k c_{ij}^k X_k, \quad i, j, k = 1, \dots, n,$$

and that in addition they span a basis of  $T_x M$  at every point  $x \in M$ . We pick up an element in the family, let us assume it is  $X_1$ , that is going to be the dynamical vector field. In order to emphasize its special rôle we will often denote it by  $\Gamma \equiv X_1$ .

The goal, therefore, is to solve the system of equations  $\dot{x}_i = \Gamma x_i$ ,  $i = 1, \dots, n$ , or, in a coordinate independent formulation, to obtain the integral curves  $\Phi_t : M \rightarrow M$  of  $\Gamma$

$$(\Gamma f)(\Phi_t(x)) = \frac{d}{dt} f(\Phi_t(x)), \quad \forall f \in C^\infty(x), \quad (5)$$

using quadratures (operations of integration, elimination and partial differentiation). The number of quadratures is given by the number of integrals of known functions depending on a finite number of parameters that are performed.

As we mentioned above  $\Gamma$  plays a distinguished and important rôle since it represents the dynamics to be integrated. In fact, and as the approach is concerned with the construction of a sequence of nested Lie subalgebras  $L_{\Gamma, k}$  of  $L$ , it will be essential that  $\Gamma$  belongs to all the subalgebras. This construction will be carried out in several steps.

The first one will be to reduce, by one quadrature, the original problem to a similar one but with a Lie subalgebra  $L_{\Gamma, 1}$  of the Lie algebra  $L$  (with  $\Gamma \in L_{\Gamma, 1}$ ) whose elements span at every point the tangent space of the leaves of a certain foliation. If iterating the procedure we end up with an Abelian Lie algebra we can, with one more quadrature, obtain the flow of the dynamical vector field (5).

We determine the foliation through a family of functions that are constant on the leaves. We first consider the ideal

$$L_{\Gamma, 1} = \langle \Gamma \rangle + [L, L], \quad \dim L_{\Gamma, 1} = n_1,$$

that, in order to make the notation simpler, we will assume to be generated by the first  $n_1$  vector fields of the family, i.e.  $L_{\Gamma, 1} = \langle \Gamma, X_2, \dots, X_{n_1} \rangle$ . This can be always achieved by choosing appropriately the basis of  $L$ .

Now take  $\zeta_1 \in L_{\Gamma, 1}^0$ , where  $L_{\Gamma, 1}^0$  is the annihilator of  $L_{\Gamma, 1}$ , i.e. the set of elements in  $L^*$  that kill all vectors of  $L_{\Gamma, 1}$ . Now we define the 1-form  $\alpha_{\zeta_1}$  on  $M$  by its action on the vector fields in  $L$  in the following way

$$\alpha_{\zeta_1}(X) = \zeta_1(X), \quad \text{for } X \in L.$$

As  $\alpha_{\zeta_1}(X)$  is a constant function on  $M$ , for any vector field in  $L$ , we have

$$d\alpha_{\zeta_1}(X, Y) = \alpha_{\zeta_1}([X, Y]) = \zeta_1([X, Y]) = 0, \quad \text{for } X, Y \in L, \quad \zeta_1 \in L_{\Gamma, 1}^0.$$

Therefore the 1-form  $\alpha_{\zeta_1}$  is closed and by application of the result of the lemma 2 the system of partial differential equations

$$X_i Q_{\zeta_1} = \alpha_{\zeta_1}(X_i), \quad i = 1, \dots, n, \quad Q_{\zeta_1} \in C^\infty(M), \quad (6)$$



has a unique (up to the addition of a constant) local solution which can be obtained by one quadrature.

For further purposes it will be convenient that the solution of (6) depends linearly on  $\zeta_1$ . This can be always achieved if we follow the construction of the lemma and we fix the same reference point  $x_0$  for any  $\zeta_1$ . In fact,  $\alpha_{\zeta_1}$  depends linearly on  $\zeta_1$  and, if  $\gamma_x$  is independent of  $\zeta_1$ , we have that the correspondence

$$L_{\Gamma,1}^0 \ni \zeta_1 \mapsto Q_{\zeta_1} \in C^\infty(M),$$

defines an injective linear map.

The previous system of equations expresses the fact that the vector fields in  $L_{\Gamma,1}$  (including  $\Gamma$ ) are tangent to

$$N_1^{[Y_1]} = \{x \mid Q_{\zeta_1}(x) = \zeta_1(Y_1), \zeta_1 \in L_{\Gamma,1}^0\} \subset M$$

for any  $[Y_1] \in L/L_{\Gamma,1}$ . Locally, for an open neighbourhood  $U$ , the  $N_1^{[Y_1]}$ 's define a smooth foliation of  $n_1$ -dimensional leaves.

Now, we repeat the previous procedure by taking  $L_{\Gamma,1}$  as the Lie algebra and any leaf  $N_1^{[Y_1]}$  as the manifold. The new subalgebra  $L_{\Gamma,2} \subset L_{\Gamma,1}$  is defined by

$$L_{\Gamma,2} = \langle \Gamma \rangle + [L_{\Gamma,1}, L_{\Gamma,1}], \quad \dim L_{\Gamma,2} = n_2,$$

and taking  $\zeta_2 \in L_{\Gamma,2}^0 \subset L_{\Gamma,1}^*$  (the annihilator of  $L_{\Gamma,2}$ ), we arrive at a new system of partial differential equations

$$X_i Q_{\zeta_2}^{[Y_1]} = \zeta_2(X_i), \quad i = 1, \dots, n_1, \quad Q_{\zeta_2}^{[Y_1]} \in C^\infty(N_1^{[Y_1]}),$$

that can be solved with one quadrature in such a way that  $Q_{\zeta_2}^{[Y_1]}$  depends linearly on  $\zeta_2$ .

For later purposes, it will be useful to extend  $Q_{\zeta_2}^{[Y_1]}$  to  $U$ . In order to do that we first introduce the map

$$U \ni x \mapsto [Y_1^x] \in L_{\Gamma,0}/L_{\Gamma,1},$$

where  $x$  and  $[Y_1^x]$  are related by the equation  $Q_{\zeta_1}(x) = \zeta_1(Y_1^x)$ , that correctly determines the map. Now, we define  $Q_{\zeta_2} \in C^\infty(U)$  by

$$Q_{\zeta_2}(x) = Q_{\zeta_2}^{[Y_1^x]}(x).$$

Note that, by construction,  $x \in N_1^{[Y_1^x]}$  and therefore the definition makes sense. It is also clear that the resulting function  $Q_{\zeta_2}(x)$  is smooth provided the reference point of the lemma changes smoothly from leave to leave; a property that can be always fulfilled.

The construction is then iterated by defining

$$N_2^{[Y_1][Y_2]} = \{x \mid Q_{\zeta_1}(x) = \zeta_1(Y_1), \quad Q_{\zeta_2}(x) = \zeta_2(Y_2), \text{ with } \zeta_1 \in L_{\Gamma,1}^0, \zeta_2 \in L_{\Gamma,2}^0\} \subset M,$$

for  $[Y_1] \in L_{\Gamma,0}/L_{\Gamma,1}$  and  $[Y_2] \in L_{\Gamma,1}/L_{\Gamma,2}$ . Note that  $L_{\Gamma,2}$  generates at every point the tangent space of  $N_2^{[Y_1][Y_2]}$ , therefore we can proceed as before.

The algorithm ends if after some steps, say  $k$ , the Lie algebra  $L_{\Gamma,k} = \langle X_1, \dots, X_{n_k} \rangle$ , whose vector fields are tangent to the  $n_k$ -dimensional leaf  $N_k^{[Y_1], \dots, [Y_k]}$ , is Abelian. In this moment the system of equations

$$X_i Q_{\zeta_k}^{[Y_1], \dots, [Y_k]} = \zeta_k(X_i), \quad i = 1, \dots, n_{k-1}, \quad Q_{\zeta_k}^{[Y_1], \dots, [Y_k]} \in C^\infty(N_k^{[Y_1], \dots, [Y_k]}),$$

can be solved locally by one more quadrature for any  $\zeta_k \in L_{\Gamma,k}^*$  (we note that, as the final Lie algebra  $L_{\Gamma,k}$  is Abelian, the integrability condition is always satisfied and we can take  $\zeta_k$  in the whole of  $L_{\Gamma,k}^*$  instead of  $L_{\Gamma,k+1}^0$ ). Then, as before, we extend the solutions to  $U$  and call them  $Q_{\zeta_k}$ .

With all these ingredients we can find the flow of  $\Gamma$  by performing only algebraic operations. In fact, consider the formal direct sum

$$\Xi = L_{\Gamma,1}^0 \oplus L_{\Gamma,2}^0 \oplus \dots \oplus L_{\Gamma,k}^0 \oplus L_{\Gamma,k}^*$$

that, as one can check, has dimension  $n$ . The linear maps  $L_{\Gamma,i}^0 \ni \zeta_i \mapsto Q_{\zeta_i} \in C^\infty(U)$  can be extended to  $\Xi$  so that to any  $\xi \in \Xi$  we assign a  $Q_\xi \in C^\infty(U)$ . Now consider a basis

$$\{\xi_1, \dots, \xi_n\} \subset \Xi.$$

The associated functions  $Q_{\xi_j}, j = 1, \dots, n$  are functionally independent and satisfy

$$\Gamma Q_{\xi_j}(x) = \xi_j(\Gamma), \quad j = 1, 2, \dots, n, \quad (7)$$

where it should be noticed that, as  $\Gamma \in L_{\Gamma,l}$  for any  $l = 0, \dots, k$ , the right hand side is well defined. From (7) we see that, in the coordinates given by  $Q_{\xi_j}(x)$ ,  $j = 1, \dots, n$ , the vector field  $\Gamma$  has constant components and, then, it is trivially integrated

$$Q_{\xi_j}(\Phi_t(x)) = Q_{\xi_j}(x) + \xi_j(\Gamma)t.$$

Now, with algebraic operations, one can derive the flow  $\Phi_t(x)$ . Altogether we have performed  $k+1$  quadratures.

## 5 Algebraic properties

The previous procedure works if it reaches an end point, i.e. if there is a smallest non negative integer  $k$  such that

$$L_{\Gamma,k} = \langle \Gamma \rangle + [L_{\Gamma,k-1}, L_{\Gamma,k-1}] \text{ for } k > 0, \quad L_{\Gamma,0} = L,$$

is an Abelian algebra. In that case we will say that  $(M, L, \Gamma)$  is *Lie integrable of order  $k+1$* .

The content of the previous section can, thus, be summarized in the following.

**Theorem 3.** If  $(M, L, \Gamma)$  is Lie integrable of order  $r$ , then the integral curves of  $\Gamma$  can be obtained by  $r$  quadratures.

We will discuss below some necessary and sufficient conditions for the Lie integrability.

**Proposition 4.** If  $(M, L, \Gamma)$  is Lie integrable, then  $L$  is solvable.

*Proof.* Denote by  $L_{(i)}$  the elements of the derived series,  $L_{(i+1)} = [L_{(i)}, L_{(i)}]$ ,  $L_{(0)} = L$ , (note that  $L_{(i)} = L_{0,i}$ ). We will show by induction that

$$L_{(i)} \subset L_{\Gamma,i}. \quad (8)$$

Clearly, this is true for  $i = 0$  and, assuming that it also holds for some  $i$ , we have the following

$$L_{(i+1)} = [L_{(i)}, L_{(i)}] \subset [L_{\Gamma,i}, L_{\Gamma,i}] \subset L_{\Gamma,i+1},$$

that completes the induction.

Then, if the system is Lie integrable, i.e.  $L_{\Gamma,k}$  is Abelian for some  $k$ , then we have  $L_{(k+1)} = 0$  and, therefore,  $L$  is solvable.  $\square$

**Proposition 5.** If  $L$  is solvable and  $A$  is an Abelian ideal of  $L$ , then  $(M, L, \Gamma)$  is Lie integrable for any  $\Gamma \in A$ .

*Proof.* Using that  $A$  is an ideal containing  $\Gamma$ , we can show that

$$A + L_{\Gamma,i} = A + L_{(i)}.$$

We proceed again by induction; if the previous holds, then

$$\begin{aligned} A + L_{\Gamma,i+1} &= A + [L_{\Gamma,i}, L_{\Gamma,i}] = A + [A + L_{\Gamma,i}, A + L_{\Gamma,i}] = \\ &= A + [A + L_{(i)}, A + L_{(i)}] = A + L_{(i+1)}. \end{aligned} \quad (9)$$

Now,  $L$  is solvable if some  $L_{(k)} = 0$  and therefore  $L_k \subset A$ , i.e. it is Abelian and, henceforth, the system is Lie integrable.  $\square$

Note that the particular case in which  $A = \langle \Gamma \rangle$  corresponds to the standard Lie theorem (Thm. 1 of section 2).

Nilpotent algebras of vector fields [14, 15] also play an interesting role in the integrability of vector fields.

**Proposition 6.** If  $L$  is nilpotent,  $(M, L, \Gamma)$  is Lie integrable for any  $\Gamma \in L$ .

*Proof.* Let us consider now the central series  $L^{(i+1)} = [L, L^{(i)}]$  with  $L^{(0)} = L$ .  $L$  nilpotent means that there is a  $k$  such that  $L^{(k)} = 0$ . Now, by induction, it is easy to see that  $L_{\Gamma,i} \subset \langle \Gamma \rangle + L^{(i)}$  and therefore  $L_{\Gamma,k} = \langle \Gamma \rangle$ . Then,  $L_{\Gamma,k}$  is Abelian and the system is Lie integrable.  $\square$

## 6 An interesting example

We now analyze the particular case of a superintegrable system studied in [16]. In this case the system is Hamiltonian, that is, the dynamical vector field  $\Gamma_H$  is obtained from a Hamiltonian function  $H$  by making use of a symplectic structure  $\omega_0$  defined in the cotangent bundle  $T^*Q$  ( $Q$  is the configuration space). Nevertheless we are now interested in considering this system just as a dynamical system (without mentioning the existence of a symplectic structure) and focusing our attention on the Lie algebra structure of the symmetries.

The dynamics is given by the vector field  $X_1 = \Gamma$ , defined in  $M = \mathbb{R}^2 \times \mathbb{R}^2$  with coordinates  $(x, y, p_x, p_y)$ , by

$$\Gamma = p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} - \frac{k_2}{y^{2/3}} \frac{\partial}{\partial p_x} + \frac{2}{3} \frac{k_2 x + k_3}{y^{5/3}} \frac{\partial}{\partial p_y},$$

where  $k_2$  and  $k_3$  are arbitrary constants. Now, with  $X_i$ ,  $i = 2, 3, 4$ , we denote the vector fields

$$\begin{aligned} X_2 &= \left( 6p_x^2 + 3p_y^2 + k_2 \frac{6x}{y^{2/3}} + k_3 \frac{6}{y^{2/3}} \right) \frac{\partial}{\partial x} + (6p_x p_y + 9k_2 y^{1/3}) \frac{\partial}{\partial y} \\ &\quad - k_2 \frac{6}{y^{2/3}} p_x \frac{\partial}{\partial p_x} + \left( 4k_2 \frac{x}{y^{5/3}} - 3 \frac{1}{y^{2/3}} p_y \right) \frac{\partial}{\partial p_y}, \\ X_3 &= \left( 4p_x^3 + 4p_x p_y^2 + \frac{8(k_2 x + k_3)}{y^{2/3}} p_x + 12k_2 y^{1/3} p_y \right) \frac{\partial}{\partial x} \\ &\quad + \left( 4p_x^2 p_y + 12k_2 y^{1/3} p_x \right) \frac{\partial}{\partial y} - 4k_2 \frac{1}{y^{2/3}} p_x^2 \frac{\partial}{\partial p_x} \\ &\quad + \left( \frac{8}{3} \frac{k_2 x + k_3}{y^{5/3}} p_x^2 - 4k_2 \frac{1}{y^{2/3}} p_x p_y - 12k_2^2 \frac{1}{y^{1/3}} \right) \frac{\partial}{\partial p_y}, \end{aligned}$$

and

$$\begin{aligned} X_4 &= \left( 6p_x^5 + 12p_x^3 p_y^2 + 24 \frac{k_3 + k_2 x}{y^{2/3}} p_x^3 + 108 k_2 y^{1/3} p_x^2 p_y + 324 k_2^2 y^{2/3} p_x \right) \frac{\partial}{\partial x} \\ &\quad + \left( 6p_x^4 p_y + 36 k_2 y^{1/3} p_x^3 \right) \frac{\partial}{\partial y} - 6 \left( \frac{k_2}{y^{2/3}} p_x^4 - 972 k_2^3 \right) \frac{\partial}{\partial p_x} \\ &\quad + \left( 4 \frac{k_3 + k_2 x}{y^{5/3}} p_x^4 - 12 \frac{k_2}{y^{2/3}} - 108 k_2^2 \frac{1}{y^{1/3}} p_x^2 \right) \frac{\partial}{\partial p_y}. \end{aligned}$$

Then, we have

- (i) The three vector fields  $X_i$  Lie commute with  $X_1 = \Gamma$

$$[\Gamma, X_i] = 0, \quad i = 2, 3, 4.$$

- (ii) The Lie brackets of the  $X_i$  between themselves are given by

$$[X_2, X_3] = 0, \quad [X_2, X_4] = 1944 k_2^3 \Gamma, \quad [X_3, X_4] = 432 k_2^3 X_2.$$

Therefore, we have the following properties. First,  $\Gamma$  and the three vector fields  $X_2, X_3, X_4$  generate a four-dimensional real Lie algebra  $L$ . Second, the derived algebra  $L_{(1)} \subset L$  is two-dimensional and Abelian because it is generated by  $\Gamma$  and  $X_2$ . Finally, the second derived algebra  $L_{(2)}$  reduces to the trivial algebra, that is,  $L_{(2)} = [L_{(1)}, L_{(1)}] = \{0\}$ . Therefore the Lie algebra  $L$  is solvable. However,  $L^{(2)} = [L, L_{(1)}]$  is not trivial but  $L^{(2)}$  is the one-dimensional ideal in  $L$  generated by  $\Gamma$ , and this implies that the Lie algebra is nilpotent. Consequently  $(M, L, X_i)$  is integrable for any index  $i$ .

## 7 Distributional integrability

The previous construction is in some sense too rigid or too restrictive. For instance, the very simple system in  $\mathbb{R}^n$  with dynamical vector field

$$\Gamma = f(x)\partial_1$$

which corresponds to the system of equations

$$\dot{x}^1 = f(x), \quad \dot{x}^2 = 0, \quad \dots, \quad \dot{x}^n = 0,$$

can be easily solved by quadratures. However, if one considers the natural choice

$$L = \langle \Gamma, \partial_2, \dots, \partial_n \rangle,$$

the vector fields do not close a Lie algebra over the real numbers. It would be worth extending the results in the previous sections to allow for Lie algebras over the ring of functions and accommodate this and other interesting cases. In the following we will pursue this goal.

In order to proceed, we shall need some preliminary definitions and results.

**Definition 1.**

For any subset  $S \subset \mathfrak{X}(M)$ , we denote by  $\mathcal{D}_S$  the  $C^\infty(M)$ -module generated by  $S$ , i.e.

$$\mathcal{D}_S = \left\{ \sum_i f^i X_i \in \mathfrak{X}(M) \mid f^i \in C^\infty(M), X_i \in S \right\}.$$

As  $\mathcal{D}_S$  is the module of sections of the corresponding generalized distribution, we will also refer to  $\mathcal{D}_S$  as to a distribution.

**Definition 2.**

We say that a real vector space  $V \subset \mathfrak{X}(M)$  is *regular* if  $V$  is isomorphic to its restriction  $V_p \subset T_p M$  at any point  $p \in M$ , and *completely regular* if it is regular and  $V_p = T_p M$ .

The previous definitions immediately imply the following.

**Proposition 7.**

- 1) Any subspace of a regular space is regular.
- 2) For any two subspaces  $W_1, W_2$  of a regular space we have  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} = \mathcal{D}_{W_1 \cap W_2}$ .

Based on these properties we introduce the following definition:

**Definition 3.** Given a completely regular space  $V \subset \mathfrak{X}(M)$  and a subset  $S \subset \mathfrak{X}(M)$ , we shall call the *core of  $S$*  in  $V$ , denoted by  $S_*$ , the smallest subspace of  $V$  such that  $S \subset \mathcal{D}_{S_*}$ .

That the previous definition makes sense and the core of any set  $S \subset \mathcal{D}_V$  exists, is contained in the following.

**Proposition 8.** For a completely regular space  $V \subset \mathfrak{X}(M)$  and any  $S \subset \mathfrak{X}(M)$ ,

$$S_* = \bigcap_{W \in \mathcal{W}} \langle W \rangle, \quad \text{with} \quad \mathcal{W} = \{W \subset V \mid S \subset \mathcal{D}_W\}.$$

Usually  $V$  will be fixed once for all and this is the reason why it does not appear in the notation.

Now, for a completely regular space  $V \in \mathfrak{X}(M)$  and a dynamical vector field  $\Gamma \in V$ , we introduce the following series:  $V_{\Gamma,0} = V$  and

$$V_{\Gamma,m} = \langle \Gamma \rangle + [V_{\Gamma,m-1}, V_{\Gamma,m-1}]_*.$$

Observe that  $V_{\Gamma,m} \subset V_{\Gamma,m-1}$ , which is easily shown by induction. If there exists a  $V_{\Gamma,k}$  for non negative  $k$  which is the first Abelian subspace in the series, we shall say that  $(M, V, \Gamma)$  is *distributionally integrable of order  $k+1$* . Then, we can state the main result of this section.

**Theorem 9.** If  $(M, V, \Gamma)$  is distributionally integrable of order  $r$ , then the vector field  $\Gamma$  can be integrated by  $r$  quadratures.

*Proof.* The procedure to obtain the solution of the system of differential equations is the same as the one sketched in the first section. For any  $\zeta_1 \in V^*$  annihilating  $V_{\Gamma,1}$ , we introduce the 1-form  $\alpha_{\zeta_1}$  such that  $\alpha_{\zeta_1}(X_i) = \zeta_1(X_i)$ . Then, one immediately sees that  $d\alpha_{\zeta_1}(X_i, X_j) = 0$  and, therefore, the 1-form is closed. Then, by applying Lemma 2, the system of partial differential equations

$$X_i Q_{\zeta_1} = \zeta_1(X_i), \quad i = 1, \dots, n, \quad Q_{\zeta_1} \in C^\infty(U),$$

has a solution obtained with one quadrature.

By construction, vector fields in  $V_{\Gamma,1}$  are tangent to the level set of the  $Q_{\zeta_1}$ 's and, then, we can reduce the problem to that submanifold.

Iterating the procedure we finally solve the problem by performing  $r$  quadratures that ends the proof.  $\square$

To illustrate this construction we will provide two examples that represent, somehow, two opposite ends.

**Example 1.** The first example was mentioned at the beginning of this section. Consider  $M = \mathbb{R}^n$ ,  $\Gamma = f(x)\partial_1$  with  $f$  being a nowhere vanishing smooth function, and

$$V = \langle \Gamma, \partial_2, \dots, \partial_n \rangle.$$

Then, we immediately see that  $[\Gamma, \partial_i] \in \mathcal{D}_{\langle \Gamma \rangle}$  for any  $i$  and therefore  $V_1 = \langle \Gamma \rangle$ , so the system of equations is solved with 2 quadratures.

**Example 2.** The second example is at the opposite end as it requires  $n$  quadratures. In this case we take

$$\Gamma = f(x)(\partial_1 + g^2(x^1)\partial_2 + \dots + g^{n-1}(x^1, \dots, x^{n-2})\partial_{n-1} + g^n(x^1, \dots, x^{n-1})\partial_n), \quad (10)$$

with  $f(x) \neq 0$  everywhere and

$$V = \langle \Gamma, \partial_2, \dots, \partial_n \rangle.$$

It is immediate to show that  $V_{\Gamma,1} = \langle \Gamma, \partial_3, \dots, \partial_n \rangle$ ,  $V_{\Gamma,2} = \langle \Gamma, \partial_4, \dots, \partial_n \rangle$ , and finally  $V_{\Gamma,n-1} = \langle \Gamma \rangle$ . This shows that the system is distributionally integrable and requires  $n$  quadratures for its solution.

Note that in the previous examples there is an arbitrary, nowhere vanishing function  $f$  that multiplies the dynamical vector field. This is, actually, the general situation as it is stated in the following proposition.

**Proposition 10.** Suppose that  $(M, V, \Gamma)$ , with  $V = \langle \Gamma, X_2, \dots, X_n \rangle$ , is distributionally integrable of order  $r$ . Then, for any nowhere-vanishing  $f \in C^\infty(M)$ , the system  $(M, V', f\Gamma)$  with  $V' = \langle f\Gamma, X_2, \dots, X_n \rangle$  is distributionally integrable of order  $r' \in \{r-1, r, r+1\}$

*Proof.* By induction, it is easy to see that if  $\{\Gamma, Y_2, \dots, Y_l\} \subset V_{\Gamma, m}$  is a basis of  $V_{\Gamma, m}$ , then  $\{f\Gamma, Y_2, \dots, Y_l\}$  forms a basis of  $V_{f\Gamma, m}$ . Therefore, if  $V_{\Gamma, r-1}$  is Abelian, then  $V_{f\Gamma, r}$  is Abelian too, i.e.  $r' \leq r+1$ . But the relation is obviously symmetric, therefore we must have  $r \leq r'+1$ , from which we get  $r-1 \leq r' \leq r+1$ .  $\square$

We can also translate the properties of the previous section to this generalized setup. We shall call a completely regular  $V$  *distributionally solvable* if the series  $V_{(i)} = [V_{(i-1)}, V_{(i-1)}]_*$ , with  $V_{(0)} = V$ , stabilizes trivially,  $V_{(n)} = \{0\}$ . Denote with  $\mathcal{D}_{(s)}$  the distribution  $\mathcal{D}_{V_{(s)}}$ . It is clear that  $\mathcal{D}_{(s)} \subset \mathcal{D}_{(s')}$  for  $s > s'$  and  $[\mathcal{D}_{(s)}, \mathcal{D}_{(s)}] \subset \mathcal{D}_{(s)}$ , so that these distributions are involutive, hence integrable.

We will say analogously that a completely regular  $V$  is *distributionally nilpotent* if the generalized central series:

$$V^{(i)} = [V^{(i-1)}, V]_*, \quad \text{with } V^{(0)} = V,$$

stabilizes at  $\{0\}$ .

**Example 3.** Consider in  $\mathfrak{X}(\mathbb{R}^n)$  the vector subspace  $V$  spanned by a basis  $X_i, i = 1, \dots, n$ , of vector fields of the following ‘triangular form’:

$$X_i = \partial_i + \sum_{k>i} f_i^k(x) \partial_k.$$

Then,  $V$  is distributionally solvable. Indeed, the vector fields in  $[V, V]$  have no  $\partial_1$  components, so  $V_{(1)}$  is spanned by  $X_2, \dots, X_n$ . Inductively,  $V_{(n-1)}$  is spanned by  $X_n = \partial_n$  and  $V_{(n)} = \{0\}$ .

**Example 4.** Consider in  $\mathfrak{X}(\mathbb{R}^n)$  the vector subspace  $V$  spanned by a basis  $X_i, i = 1, \dots, n$ , of vector fields of the following ‘strong triangular form’:

$$X_i = \partial_i + \sum_{k>i} f_i^k(x^1, \dots, x^{k-1}) \partial_k,$$

where the coefficients  $f_i^k$  depend on variables  $x^1, \dots, x^{k-1}$  only. Then,  $V$  is distributionally nilpotent. Indeed, as before  $V^{(1)}$  is spanned by  $X_2, \dots, X_n$  and, inductively,  $[V, V^{(s)}]$  is spanned by  $X_{s+1}, \dots, X_n$ , so  $V^{(n)} = \{0\}$ .

The proof of the following proposition is completely analogous to the proofs of Proposition 4 and Proposition 5.

**Proposition 11.**

- (a) If  $(M, V, \Gamma)$  is distributionally integrable, then  $V$  is distributionally solvable.

- (b) If  $V$  is distributionally nilpotent, then  $(M, V, \Gamma)$  is distributionally integrable for any  $\Gamma \in V$ .

We end up with presenting descriptions of distributionally solvable or nilpotent  $V \subset \mathfrak{X}(M)$  showing that the Examples 3 and 4 are in a sense universal (compare with [14, 15]).

Let  $V \subset \mathfrak{X}(M)$  be distributionally solvable,  $r$  be the smallest natural number such that  $V_{(r)} = \{0\}$ , and  $d_s$ ,  $s = 1, \dots, r$ , be the dimension of the space  $V_{(s-1)}/V_{(s)}$ . Put  $w_s = d_1 + \dots + d_s$ , for  $s \geq 1$ , to be the dimension of  $V/V_{(s)}$ .

**Theorem 12.** For any  $p \in M$ , there is a basis  $\{X_1, \dots, X_n\}$  of  $V$  and there exist local coordinates  $(x^u)$  around  $p$  such that, for  $w_{s-1} < i \leq w_s$ ,  $s = 1, \dots, r$ , the vector field  $X_i$  is of the form

$$X_i = \partial_i + \sum_{k > w_s} f_i^k(x) \partial_k.$$

*Proof.* Choose  $X_1, \dots, X_{d_1}$  representing a basis of  $V/V_{(1)}$ . Let  $\alpha^1, \dots, \alpha^{d_1}$  be 1-forms on  $M$  such that  $\alpha^j(X_i) = \delta_i^j$  and  $\alpha^j(V_{(1)}) = 0$ , for  $i, j = 1, \dots, w_1$ . Exactly as we have seen earlier, the 1-forms are closed and they have potentials  $x^i$  around  $p$  of the form

$$x^i(q) = \int_{\gamma_q} \alpha^i,$$

where  $\gamma_q$  is a smooth path joining  $p$  with  $q \in M$ .

Now we can choose  $X_{d_1+1}, \dots, X_{w_2} \in V_{(1)}$  representing a basis of  $V_{(1)}/V_{(2)}$ . As the distribution generated by  $V_{(1)}$  is clearly involutive, it defines a foliation by the level sets  $M(a)$  of  $(x^1, \dots, x^{d_1})$ , so we can choose a smooth submanifold  $N_1$  of dimension  $d_1$  through  $p$  intersecting the local leaf  $M(a)$  of  $V_{(1)}$ , where  $a \in \mathbb{R}^{d_1}$  is sufficiently close to 0, at a single point  $p(a)$ .

Inductively, for any such  $a$ , we have on  $M(a)$  the 1-forms  $\alpha_a^{d_1+1}, \dots, \alpha_a^{w_2}$  such that  $\alpha^j(X_i) = \delta_i^j$  and  $\alpha^j(V_{(2)}) = 0$ , for  $i, j = d_1 + 1, \dots, w_2$ . The 1-forms are, this time leaf-wise, closed and they have potentials  $x^i$  around  $p$ ,  $i = d_1 + 1, \dots, w_2$ , of the form

$$x^i(q) = \int_{\gamma_q(a)} \alpha^i, \tag{11}$$

where  $\gamma_q(a)$  is a smooth path lying entirely inside the leaf  $M(a)$  and joining  $q \in M(a)$  with the point  $p(a)$ . The functions  $x^i$ , with  $i = d_1 + 1, \dots, w_2$ , are globally defined and smooth, and they are, this time only leaf-wise, potentials for  $\alpha^i$ . In any case,  $X_i(x^j) = \delta_i^j$  and  $V_{(2)}(x^j) = 0$ , for  $i, j = d_1 + 1, \dots, w_2$ .

Proceeding in this way inductively, we prove the theorem.  $\square$

If we assume that  $V \subset \mathfrak{X}(M)$  is distributionally nilpotent and define  $R$ ,  $D_s$ ,  $W_s$  like  $r$ ,  $d_s$ , and  $w_s$  above, but using the sequence of subspaces  $V^{(s)}$  instead of  $V_{(s)}$ , we get a basis  $(X_i)$  of  $V$  and local coordinates  $x^i$  as before, with

$$X_i = \partial_i + \sum_{k > W_s} f_i^k(x) \partial_k,$$

for any  $i$  satisfying  $W_{s-1} < i \leq W_s$ ,  $s = 1, \dots, R$ . This time, however,  $[X_i, V^{(s')}]$  lies in the distribution spanned by  $V^{(s'+1)}$ . This means, for  $s' > s$ , that  $\partial_j(f_i^k) = 0$  for  $k \leq W_{s'}$  and  $j > W_{s'-1}$ , thus we get a stronger triangular form for elements of  $V$  as follows.



**Theorem 13.** Let  $V \subset \mathfrak{X}(M)$  be distributionally nilpotent,  $R$  be the smallest natural number such that  $V^{(R)} = \{0\}$ , and  $W_s$  be the dimension of  $V/V^{(s)}$ . Then, for any  $p \in M$ , there is a basis  $X_1, \dots, X_n$  of  $V$  and local coordinates  $(x^u)$  around  $p$  such that, for  $W_{s-1} < i \leq W_s$ ,  $s = 1, \dots, R$ , the vector field  $X_i$  is of the form

$$X_i = \partial_i + \sum_{k > W_s} f_i^k(x) \partial_k,$$

where all coefficients  $f_i^k$ , with  $k \leq W_{s'+1}$ ,  $s \leq s'$ , depend on variables  $x^1, \dots, x^{W_{s'}}$  only.

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